ON NILPOTENT COINVARIANT MODULES

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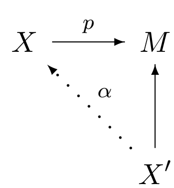
Abstract: A module is called nilpotent-coinvariant if it is invariant under any nilpotent endomorphism of its projective cover. Some properties of this class of these modules were investigated by Truong Cong Quynh, Adel Abyzov, Dinh Duc Tai [8]. In this paper, we continue to study nilpotent-coinvariant modules over general covers.

Keywords: Nilpotent-coinvariant; projective cover; $\mathcal{X}$-(pre)cover.

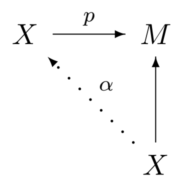
1 Introduction

Throughout this article, all rings are associative rings with identity and all modules are right unital. For a submodule N of M, we use $N \leq M$ ($N < M$) to mean that N is a submodule of M (respectively, proper submodule). We usually write $\mathrm{End}(M)$ ($\mathrm{Aut}(M)$) to indicate its ring of right $R-$module endomorphisms (respectively, automorphism). A submodule N of a module M is called small in M (denoted as $N \ll M$) if $N + K \neq M$ for any proper submodule K of M.

Let $\mathcal{X}$ be a class of R-modules closed under isomorphisms. An R-homomorphism $p : X \to M$ is an $\mathcal{X}$-precover of a module M provided that $X \in \mathcal{X}$ and each diagram



with $X' \in \mathcal{X}$ can be completed by a homomorphism $\alpha : X' \to X$ to a commutative diagram. If, moreover, the diagram



can be only completed by automorphisms α, we call X an $\mathcal{X}$-cover of M.

In [3], Guil Asensio, Keskin and Srivastava introduced the notion of $\mathcal{X}$-automorphism-coinvariant modules. It shows that the endomorphism ring of an $\mathcal{X}$-automorphism-coinvariant module is a semiregular ring. Some other properties of $\mathcal{X}$-automorphism-coinvariant modules are studied.

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A right $R-$module M having an $\mathcal{X}-$cover $p: X \to M$ is said to be $\mathcal{X}-$endomorphism-coinvariant ($\mathcal{X}-$ automorphism-coinvariant) if for any endomorphism (resp., automorphism) g of X, there exists an endomorphism f of M such that $f \circ p = p \circ g$.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & M \end{array}$$

In this paper, we introduce the notion of $\mathcal{X}$-nilpotent-coinvariant modules with accompanying conditions and study some properties of them. It shows that if M is an $\mathcal{X}-$nilpotent-coinvariant strongly $\mathcal{X}-$copure module then M a $D3$-module.

2 On $\mathcal{X}$-nilpotent-coinvariant

Theorem 2.1. *If $p: X \to M$ and $p': X' \to M$ are two $\mathcal{X}-$covers of a right $R-$module M then $X' \cong X$.*

Proof. By Theorem 1.2.6 in [4]. □

Theorem 2.2. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a direct sum of submodules M_i of M and let $p_i : X_i \to M_i$ be $\mathcal{X}-$covers of M_i. Then, $\oplus p_i : \oplus X_i \to M$ is an $\mathcal{X}-$cover of M.*

Proof. By Theorem 1.2.10 in [4]. □

Definition 2.3. A right $R-$module M having an $\mathcal{X}-$cover $p : X \to M$ is said to be $\mathcal{X}-$nilpotent-coinvariant if for any nilpotent endomorphism g of X, there exists an endomorphism f of M such that $f \circ p = p \circ g$.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & M \end{array}$$

From the definitions of endomorphism(automorphism-coinvariant)-coinvariant modules, we have the following:

$\mathcal{X}-$endomorphism-coinvariant $\Rightarrow$ $\mathcal{X}-$automorphism-coinvariant$\Rightarrow$ $\mathcal{X}-$nilpotent-coinvariant

Theorem 2.4. *Let M be a right $R-$module with an epimorphic $\mathcal{X}-$cover $p : X \to M$. Then M is $\mathcal{X}-$nilpotent-coinvariant if and only if $g(ker(p)) \leq ker(p)$ for any nilpotent endomorphism g of X.*

Proof. ($\Rightarrow$) Assume that M is $\mathcal{X}-$nilpotent-coinvariant. Then, for any nilpotent endomorphism g of X, there exists an endomorphism f of M such that $p \circ g = f \circ p$. Hence, $(p \circ g)(\mathrm{ker}(p))$=$f(p(\mathrm{ker}(p))) = 0$, and so $g(\mathrm{ker}(p)) \leq \mathrm{ker}(p)$.

($\Leftarrow$) We consider the following homomorphism: $\varphi : X/ker(p) \to M$, $\varphi(x + ker(p)) = p(x) + ker(p)$. It is easy to see that φ is well-defined. Since p is an epimorphism, φ is an isomorphism and we have $\varphi \circ \pi = p$, where π is the canonical projection.

$$\begin{array}{ccc} X & \xrightarrow{p} & M \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \varphi} & \\ X/ker(p) & & \end{array}$$

Let g be an arbitrary nilpotent endomorphism of X. Consider $h : X/ker(p) \to X/ker(p), h(x+ker(p)) = g(x) + ker(p)$. Since $g(\ker(p)) \leq \ker(p)$, then h is well-defined. It is clear that we have $\pi \circ g = h \circ \pi$.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X/ker(p) & \xrightarrow{h} & X/ker(p) \end{array}$$

Take $f = \varphi h \varphi^{-1} \in End(M)$. We have that $\varphi : X/ker(p) \to M$ is an isomorphism and obtain that the following diagram is commutative

$$\begin{array}{ccc} X/ker(p) & \xrightarrow{h} & X/ker(p) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ M & \xrightarrow{f} & M \end{array}$$

Now, we have $p \circ g = \varphi \circ \pi \circ g = \varphi \circ h \circ \pi = f \circ \varphi \circ \pi = f \circ p$. We deduce that M is $\mathcal{X}-$nilpotent-coinvariant. □

Definition 2.5. Let M, N be modules. We will say that N is $\mathcal{X} - M-$projective if there exist $\mathcal{X}-$covers $p_N : X_N \to N$, $p_M : X_M \to M$ satisfying that for any homomorphism $g : X_N \to X_M$, there is a homomorphism $f : N \to M$ such that $p_M g = f p_N$:

$$\begin{array}{ccc} X_N & \xrightarrow{g} & X_M \\ \downarrow{\scriptstyle p_N} & & \downarrow{\scriptstyle p_M} \\ N & \xrightarrow{f} & M \end{array}$$

If M is $\mathcal{X} - M-$projective, then M is said to be an $\mathcal{X}-$endomorphism coinvariant module.

Theorem 2.6. *Let $\mathcal{X}$ be a covering class. If N is $\mathcal{X} - M-$projective, then N' is $\mathcal{X} - M'-$projective for any direct summand N' of N and any direct summand M' of M.*

Proof. Let $N = N' \oplus K, M = M' \oplus L$ for some submodules K of N and L of M. We have $X_N = X_{N'} \oplus X_K$ is $\mathcal{X}-$cover of N and $X_M = X_{M'} \oplus X_L$ is $\mathcal{X}-$cover of M. Let $\alpha : X_{N'} \to X_{M'}$ be any homomorphism. Let $\pi : X_{N'} \oplus X_K \to X_{N'}$ be the canonical projection and $i : X_{M'} \to X_{M'} \oplus X_L$ be the inclusion map. Set $g = i\alpha\pi : X_{N'} \oplus X_K \to X_{M'} \oplus X_L$. Since N is $\mathcal{X} - M-$ projective, there exists a homomorphism $f : N \to M$ such that

$$f \circ (p_{N'} \oplus p_K) = (p_{M'} \oplus p_L) \circ g.$$

$$\begin{array}{ccc} X_N & \xrightarrow{g} & X_M \\ \Big\downarrow{\scriptstyle p_{N'}\oplus p_K} & & \Big\downarrow{\scriptstyle p_{M'}\oplus p_L} \\ N & \xrightarrow{f} & M \end{array}$$

It follows that $f \circ p_{N'} = p_{M'} \circ g$. Let $\pi_{M'} : M' \oplus L \to M'$ be the canonical projection and $i_{N'} : N' \to N' \oplus K$ be the inclusion map. Let $g' = \pi_{M'} \circ f \circ i_{N'} : N' \to M'$, then we can check that $g' \circ p_{N'} = p_{M'} \circ \alpha$.

$$\begin{array}{ccc} X_{N'} & \xrightarrow{\alpha} & X_{M'} \\ \Big\downarrow{\scriptstyle p_{N'}} & & \Big\downarrow{\scriptstyle p_{M'}} \\ N' & \xrightarrow{g'} & M' \end{array}$$

We deduce that N' is $\mathcal{X} - M'-$projective. □

Corollary 2.7. *If N is an $\mathcal{X} - M-$projective module and L is a direct summand of M, then N is an $\mathcal{X} - L-$projective module.*

Corollary 2.8. *Every direct summand of an $\mathcal{X} - M-$projective module is also an $\mathcal{X} - M-$projective module.*

Corollary 2.9. *Any direct summand of an $\mathcal{X}-$endomorphism-coinvariant module is $\mathcal{X}-$endomorphism coinvariant.*

Lemma 2.10. *Let $M = M_1 \oplus M_2$ be an $\mathcal{X}-$nilpotent-coinvariant module. Then M_1 is $\mathcal{X} - M_2-$projective.*

Proof. Let $p_1 : X_1 \to M_1; p_2 : X_2 \to M_2$ be $\mathcal{X}-$covers of M_1, M_2, respectively. Thus, $p = p_1 \oplus p_2 : X = X_1 \oplus X_2 \to M$ is an $\mathcal{X}-$cover of M. For any homomorphism $g : X_1 \to X_2$, $\bar{g} : X \to X$ with $\bar{g}(x_1 + x_2) = g(x_1)$ is a nilpotent endomorphism of X. Since X is an $\mathcal{X}-$nilpotent-coinvariant module, there exists $h : M \to M$ such that $p\bar{g} = hp$. Let $f = \pi_2 h i_1$, where $\pi_2 : M \to M_2$ is the canonical projection and $i_1 : M_1 \to M$ is the inclusion map, then we have $fp_1 = p_2 g$

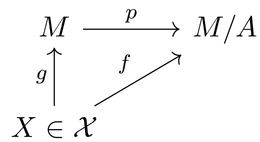
$$\begin{array}{ccc} X_1 & \xrightarrow{g} & X_2 \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle p_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

Therefore, M_1 is $\mathcal{X} - M_2-$projective. □

Corollary 2.11. *Assume that $M = M_1 \oplus M_2$. If M is $\mathcal{X}-$endomorphism coinvariant, then M_1 is $\mathcal{X} - M_2-$projective and M_2 is $\mathcal{X} - M_1-$projective.*

The two right $R-$modules M_1 and M_2 are mutually $\mathcal{X}-$projective provided that M_1 is $\mathcal{X} - M_2-$projective and M_2 is $\mathcal{X} - M_1-$projective.

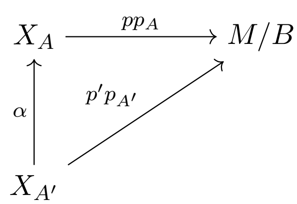
Definition 2.12. A module M is called *a strongly $\mathcal{X}-$copure module* if every submodule A of M and any homomorphism $f : X \to M/A$, $X \in \mathcal{X}$, f lifts to a homomorphism $g : X \to M$, i.e., $p \circ g = f$.



A module M is called a *D*3*-module* if for two direct summands A and B of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

Theorem 2.13. *Let $\mathcal{X}$ be a covering class and M be an $\mathcal{X}-$nilpotent-coinvariant module. If M is a strongly $\mathcal{X}-$copure module, then M a $D3$-module.*

Proof. Let A, B be direct summnands of M such that $A + B = M$. Let $M = A \oplus A'$ and $p : A \to M/B, p(a) = a + B$; $p' : A' \to M/B, p'(a') = a' + B$. Let $p_A : X_A \to A; p_{A'} : X_{A'} \to A'$ be $\mathcal{X}-$covers of A, A', respectively. Since M is a strongly $\mathcal{X}-$copure module, A is also. It follows that $pp_A : X_A \to M/B$ is a precover of M/B. There exists a homomorphism $\alpha : X_{A'} \to X_A$ such that $pp_A\alpha = p'p_{A'}$



By Lemma 2.10, there exists a homomorphism $f : A' \to A$ such that $p_A\alpha = fp_{A'}$. Therefore, $pp_A\alpha = pfp_{A'} = p'p_{A'}$. As $p_{A'}$ is an epimorphism, $pf = p'$. For all $a' \in A'$, we have

$$(pf - p')(a') = pf(a') - p'(a') = f(a') - a' + B = 0.$$

It follows that $(1 - f)(A') \leq B$. Now, for all $m \in M$, we have

$$m = a + a' = a + f(a') + (1 - f)(a').$$

Therefore $M = A + (1 - f)(A')$. For all $x \in A \cap (1 - f)(A'), x \in (1 - f)(A')$, and so there exists $x' \in A'$ such that $x = x' - f(x')$. Then, $x' = x + f(x') \in A$, and so $x' = 0$. It follows that $x = 0$. Hence $M = A \oplus (1 - f)(A')$. We have

$$B = B \cap M = B \cap [A \oplus (1 - f)(A')] = A \cap B \oplus (1 - f)(A')$$

Thus, $A \cap B$ is a direct summand of B, and therefore it is a direct summand of M. It means that M is a $D3$-module. □

In [2], a right $R-$module M is *$\mathcal{X}-$idempotent-coinvariant* provided that there exists an $\mathcal{X}-$cover $p : X \to M$ such that for every idempotent $g \in End(X)$, there is an endomorphism $f : M \to M$ such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow p & & \downarrow p \\ M & \xrightarrow{f} & M \end{array}$$

A right $R-$module M is an $\mathcal{X}-$*lifting* module provided that there is an $\mathcal{X}-$cover $p : X \to M$ of M such that for every idempotent $g \in End(X)$, there is an idempotent $f : M \to M$ such that $g(X) + ker(p) = p^{-1}(f(M))$ (see [2]).

A ring R is called *nil-clean* if any $x \in R$ can write the form $x = e+n$ for some idempotent $e \in R$ and nilpotent $n \in R$.

Theorem 2.14. *Let M be an $\mathcal{X}-$lifting module and let $p : X \to M$ be an epimorphism $\mathcal{X}-$cover with superfluous kernel. Assume that $End(X)$ is a nil-clean ring. If M is an $\mathcal{X}-$nilpotent-coinvariant module then M is an $\mathcal{X}-$endomorphism-coinvariant module.*

Proof. Let g be any endomorphism of X. By the definition of nil-clean rings, we have $g = e+u$ where e is an idempotent endomorphism of X and u is nilpotent endomorphism of X. Since M be an $\mathcal{X}-$nilpotent-coinvariant module, there exists a homomorphism $f : M \to M$ such that $pu = hp$.
On the other hand, since M is an $\mathcal{X}-$lifting module, $A = p(e(X))$ and $B = p((1-e)(X))$ are direct summands of M and $A + B = M$. By Theorem 2.13, $A \cap B$ is direct summands of M. We write $M = (A \cap B) \oplus C$ for some submodule C of M. Take $A' = A \cap C \leq A$. Then, we have $A' \oplus B = M$. For every $x \in X$, since p is epimorphic, there exists $x_1, x_2 \in X$ such that

$$p(x) = pe(x_1) + p(1-e)(x_2) = pe(x) + p(1-e)(x).$$

Then

$$p(e(x_1) + (1-e)(x_2) - e(x) + (1-e)(x)) = 0,$$

so $e(x_1) - e(x) = e(y), y \in ker(p)$. Let $\pi : A' \oplus B \to A'$ be the canonical projection. Now, we have

$$(\pi p - pe)(x) = pe(x_1) - pe(x) = pe(y) = -(\pi p - pe)(y).$$

Thus, $x + y \in ker(\pi p - pe)$, so $X = ker(p) + ker(\pi p - pe)$. Since $ker(p) \ll X, \pi p = pe$. Let $f = \pi + h \in End(M)$, then

$$pg = p(e+u) = pe + pu = \pi p + hp = (\pi + h)p = fp.$$

That means M is an $\mathcal{X}-$endomorphism coinvariant module.

□

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TÓM TẮT

LỚP CÁC MÔĐUN ĐỐI BẤT BIẾN LŨY LINH

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Ngày nhận bài 10/9/2021, ngày nhận đăng 26/10/2021

Một môđun được gọi là đối bất biến lũy linh nếu nó bất biến dưới mọi tự đồng cấu lũy linh của bao xạ ảnh chính nó. Một số tính chất của lớp môđun này đã được Trương Công Quỳnh, Adel Abyzov và Đinh Đức Tài nghiên cứu và giới thiệu trong [8]. Trong bài báo này, chúng tôi giới thiệu một số kết quả khác về lớp môđun này.

Từ khóa: Đối bất biến lũy linh; bao xạ ảnh; $\mathcal{X}$-(pre) bao.